

LONG TIME DYNAMICS FOR SEMIRELATIVISTIC NLS AND HALF WAVE IN ARBITRARY DIMENSION

JACOPO BELLAZZINI, VLADIMIR GEORGIEV, AND NICOLA VISCIGLIA

ABSTRACT. We consider the Cauchy problems associated with semirelativistic NLS (sNLS) and half wave (HW). In particular we focus on the following two main questions: local/global Cauchy theory; existence and stability/instability of ground states. In between other results, we prove the existence and stability of ground states for sNLS in the L^2 supercritical regime. This is in sharp contrast with the instability of ground states for the corresponding HW, which is also established along the paper, by showing an inflation of norms phenomenon. Concerning the Cauchy theory we show, under radial symmetry assumption the following results: a local existence result in H^1 for energy subcritical nonlinearity and a global existence result in the L^2 subcritical regime.

The aim of this paper is the analysis of the following Cauchy problems with special emphasis to the local/global existence and uniqueness results, as well as to the issue of existence and stability/instability of ground states:

$$(0.1) \quad \begin{cases} i\partial_t u = Au - u|u|^{p-1}, & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0, x) = f(x) \in H^s(\mathbb{R}^n), \end{cases}$$

where $A = \sqrt{-\Delta}$ and $A = \sqrt{1 - \Delta}$, namely Half Wave (HW) and semirelativistic NLS (sNLS). Since now on $H^s(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n)$ denote respectively the usual inhomogeneous and homogeneous Sobolev spaces in \mathbb{R}^n , endowed with the norms $\|(1 - \Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}$ and $\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}$. We shall also refer to $H_{rad}^s(\mathbb{R}^n)$ as to the set of functions belonging to $H^s(\mathbb{R}^n)$ which are radially symmetric.

Along the paper we shall study several properties of the Cauchy problems associated with sNLS and HW. The first result will concern the local/global Cauchy theory at low regularity under an extra radiality assumption. We point out that at the best of our knowledge in the literature there exist very few results about the global existence of solutions to both HW and sNLS. In particular we mention the result in [13] where it is considered HW in $1 - d$ with nonlinearity $u|u|^3$ and initial data in $H^1(\mathbb{R})$, without any further symmetry assumption. Indeed the aforementioned result can be extended to $1 - d$ sNLS with quartic nonlinearity. We also underline that in $1 - d$ no results are available concerning the global existence for higher order nonlinearity, namely $p > 4$. One novelty in this paper is that we provide global existence results in higher dimension $n \geq 2$ under the

radial symmetry assumption, provided that p satisfies some restrictions.

Another important issue considered along this article is the the existence and stability/instability properties of solitary waves associated with sNLS and HW. Of course, the first main ingredient in order to speak about dynamical properties of the solitary waves, is a robust Cauchy theory that at the best of our knowledge is provided in this paper for the first time in the radially symmetric setting.

We recall that two values of the nonlinearity p are quite relevant: the nonlinearity $u|u|^{2/(n-1)}$, which is $H^{1/2}(\mathbb{R}^n)$ -critical, and the nonlinearity $u|u|^{2/n}$, which is $L^2(\mathbb{R}^n)$ -critical. Next we present our main result about the Cauchy problems (0.1): we prove on one hand a local existence result in $H_{rad}^1(\mathbb{R}^n)$ via contraction argument for $H^{1/2}(\mathbb{R}^n)$ subcritical nonlinearity; on the other hand we show that the solutions are global in time provided that the nonlinearity is $L^2(\mathbb{R}^n)$ -subcritical and we assume an a-priori bound on $H^{1/2}(\mathbb{R}^n)$ norm of the solution.

Theorem 0.1. *Let $n \geq 2$, A be either $\sqrt{-\Delta}$ or $\sqrt{1-\Delta}$, $p \in (1, 1 + \frac{2}{n-1})$. Then for every $R > 0$ there exists $T = T(R) > 0$ and a Banach space X_T such that:*

- $X_T \subset \mathcal{C}([0, T]; H_{rad}^1(\mathbb{R}^n))$;
- *for any $f(x) \in H_{rad}^1(\mathbb{R}^n)$ with $\|f\|_{H^1(\mathbb{R}^n)} \leq R$, there exists a unique solution $u(t, x) \in X_T$ of (0.1).*

Assume moreover that $p \in (1, 1 + \frac{2}{n})$, then the solution is global in time.

We point out that by a cheap argument (based only on Sobolev embedding and energy estimates) one can solve locally in time the Cauchy problem (0.1) for initial data $f(x) \in H^{n/2+\epsilon}(\mathbb{R}^n)$ (without any radially assumption). Notice that we provide a local existence result, in radial symmetry, with regularity $H^1(\mathbb{R}^n)$ for $n \geq 2$. Indeed it will be clear to the reader, by looking at the proof of Theorem 0.1, that one can push the local theory at the level of regularity $H_{rad}^{1/2+\epsilon}(\mathbb{R}^n)$. The main technical difficulty to go from $H_{rad}^1(\mathbb{R}^n)$ to $H_{rad}^{1/2+\epsilon}(\mathbb{R}^n)$ being the fact that in the first case we work with straight derivatives, and hence the weighted chain rules that we need along the proof are straightforward. In the second case the proof requires more delicate commutator estimates that we prefer to skip along this paper. We also underline that in the L^2 subcritical regime we get a global existence result.

Next we shall analyze the issue of standing waves. We recall that standing waves are special solutions to (0.1) with a special structure, namely $u(t, x) = e^{i\omega t}v(x)$, where $\omega \in \mathbb{R}$ plays the role of the frequency. Indeed $u(t, x)$ is a standing wave solution if and only if $v(x)$ satisfies

$$(0.2) \quad Av + \omega v - v|v|^p = 0 \quad \text{in } \mathbb{R}^n.$$

It is worth mentioning that, following the pioneering paper [6], it is well understood how to build up solitary waves for both sNLS and HW via a energy constrained minimization argument, in the case of L^2 -subcritical nonlinearity. Moreover as a byproduct of this variational approach, the corresponding solitary waves are orbitally stable. We recall that sNLS and HW enjoy respectively the conservation of the following energy:

$$(0.3) \quad \mathcal{E}_s(u) = \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1},$$

$$(0.4) \quad \mathcal{E}_{hw}(u) = \frac{1}{2} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 + \frac{1}{2} \|u\|_2^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1},$$

as well as the conservation of the mass, namely:

$$(0.5) \quad \frac{d}{dt} \|u(t, x)\|_{L^2(\mathbb{R}^n)}^2 = 0$$

for solutions $u(t, x)$ associated with (0.1). In the nonlocal context in which we are interested in, the minimization problems analogue of the one studied in [6] for NLS are the following ones:

$$(0.6) \quad \mathcal{J}_r^s = \inf_{u \in S_r} \mathcal{E}_s(u), \quad \mathcal{J}_r^{hw} = \inf_{u \in S_r} \mathcal{E}_{hw}(u)$$

where

$$(0.7) \quad S_r = \{u \in H^{1/2}(\mathbb{R}^n) \text{ s.t. } \|u\|_{L^2(\mathbb{R}^n)}^2 = r\}.$$

Indeed it is not difficult (following the rather classical concentration-compactness argument, see for instance [2] for more details in the non-local setting) to get a strong compactness property (up to translation) for minimizing sequences associated with the minimization problems above, provided that the nonlinearity is L^2 subcritical, i.e. $1 < p < 1 + \frac{2}{n}$. By combining this fact with the global existence result stated in Theorem 0.1, one can prove a stability result that we state below. In order to do that first we need to introduce a suitable notion of stability, that is weaker respect to the usual one. This is due mainly to the fact that we are not able to get any global existence result for the Cauchy problem associated with sNLS and HW at the level of regularity of the Hamiltonian $H^{1/2}$ and without the radiality assumption. Hence we need to assume more regularity and also the radial symmetry on the perturbations allowed along the definition of stability.

Definition 0.1. Let $\mathcal{N} \subset H_{rad}^1(\mathbb{R}^n)$ be bounded in $H^{1/2}(\mathbb{R}^n)$. We say that \mathcal{N} is *weakly orbitally stable* by the flow associated with sNLS (resp. HW) if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} & \text{dist}_{H^{1/2}}(u(0, \cdot), \mathcal{N}) < \delta \text{ and } u(0, x) \in H_{rad}^1(\mathbb{R}^n) \Rightarrow \\ & \Phi_t(u(0, \cdot)) \text{ is globally defined and } \sup_t \text{dist}_{H^{1/2}}(\Phi_t(u(0, \cdot)), \mathcal{N}) < \epsilon \end{aligned}$$

where $dist_{H^{1/2}}$ denotes the usual distance with respect to the topology of $H^{1/2}$ and $\Phi_t(u(0, \cdot))$ is the unique global solution associated with the Cauchy problem sNLS (resp. HW) and with initial condition $u(0, x)$.

We can now state the next result, where we use the notations (0.6) and (0.7). We state it as a corollary since it is a classical consequence of the concentration-compactness argument in the spirit of [6] and Theorem 0.1, that guarantees a global dynamic for sNLS and HW. Hence we shall not provide the straightforward proof along the paper. Nevertheless we believe that it has its own interest.

Corollary 0.1. *Let $1 < p < 1 + \frac{2}{n}$ and $n \geq 1$. Then for every $r > 0$ we have:*

- $\mathcal{J}_r^s > -\infty$ (resp. $\mathcal{J}_r^{hw} > -\infty$) and $\mathcal{B}_r^s \neq \emptyset$ (resp. $\mathcal{B}_r^{hw} \neq \emptyset$) where

$$\mathcal{B}_r^s := \{v \in S_r \text{ s.t. } \mathcal{E}_s(v) = \mathcal{J}_r^s\}$$

(resp. $\mathcal{B}_r^{hw} := \{v \in S_r \text{ s.t. } \mathcal{E}_{hw}(v) = \mathcal{J}_r^{hw}\}$). In particular for every $v \in \mathcal{B}_r^s$ (resp. $v \in \mathcal{B}_r^{hw}$) there exists $\omega \in \mathbb{R}$ such that

$$\sqrt{1 - \Delta}v + \omega v - v|v|^{p-1} = 0$$

(resp. $\sqrt{-\Delta}v + \omega v - v|v|^{p-1} = 0$);

- the set \mathcal{B}_r^s (resp. \mathcal{B}_r^{hw}) is weakly orbitally stable by the flow associated with sNLS (resp. HW). Moreover in the case $n = 1$ the weak orbital stability property can be strengthened, in the sense that in the Definition 0.1 we can replace $H_{rad}^1(\mathbb{R})$ by the larger space $H^1(\mathbb{R})$.

On the contrary, the situation dramatically changes in the L^2 -supercritical regime (namely $p > 1 + \frac{2}{n}$) since the aforementioned minimization problems (0.6) are meaningless, in the sense that:

$$(0.8) \quad \mathcal{J}_r^s = \mathcal{J}_r^{hw} = -\infty, \quad \forall r > 0, \quad p > 1 + \frac{2}{n}.$$

Next result is aimed to show a special geometry (local minima) for the constrained energy associated to sNLS in the L^2 -supercritical regime, i.e $1 + \frac{2}{n} < p < 1 + \frac{2}{n-1}$. In order to state our next result let us first introduce a family of localized and constrained minimization problems:

$$(0.9) \quad \mathcal{J}_r = \inf_{u \in S_r \cap B_1} \mathcal{E}_s(u),$$

where

$$B_\rho = \{u \in H^{1/2}(\mathbb{R}^n) \text{ s.t. } \|(1 - \Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R}^n)} \leq \rho\}.$$

We also recall that the notion of *weak orbital stability* is given in Definition 0.1.

Theorem 0.2. *Let $1 + \frac{2}{n} < p < 1 + \frac{2}{n-1}$ and $n \geq 1$. There exists $r_0 > 0$ such that the following conditions occur for every $r \in (0, r_0)$:*

- $\mathcal{J}_r > -\infty$, $\mathcal{B}_r \neq \emptyset$ and $\mathcal{B}_r \subset B_{1/2} \cap H^1(\mathbb{R}^n)$, where

$$\mathcal{B}_r := \{v \in S_r \cap B_1 \text{ s.t. } \mathcal{E}_s(v) = \mathcal{J}_r\}.$$

In particular for every $v \in \mathcal{B}_r$ there exists $\omega \in \mathbb{R}$ such that

$$\sqrt{1 - \Delta}v + \omega v - v|v|^{p-1} = 0;$$

- the elements in \mathcal{B}_r are ground states on S_r , namely:

$$\inf_{\mathcal{C}_r} \mathcal{E}_s(w) = \mathcal{J}_r \text{ where } \mathcal{C}_r = \{w \in S_r \text{ s.t. } \mathcal{E}'_s|_{S_r}(w) = 0\}.$$

Assume moreover the following assumption:

$$(0.10) \quad \sup_{(-T_-(f), T_+(f))} \|u(t, x)\|_{H^{1/2}(\mathbb{R}^n)} < \infty \Rightarrow T_{\pm}(f) = \infty$$

where $(-T_-(f), T_+(f))$ is the maximal time of existence of $u(t, x)$ which is the nonlinear solution to sNLS with initial datum $f(x) \in H_{rad}^1(\mathbb{R}^n)$. Then we get:

- the set $\mathcal{B}_r \cap H_{rad}^{1/2}(\mathbb{R}^n)$ is weakly orbitally stable for the flow associated with sNLS.

We point out that the extra assumption (0.10) it is satisfied for $n = 1$ and $p = 4$ without any radiality assumption (it follows by a suitable adaptation to sNLS of the argument given in [13] for HW). An alternative and simpler argument for the global existence of $1 - d$ quartic sNLS is given in the Appendix. Hence the statement above provides the existence of stable standing waves for the quartic $1 - d$ sNLS, by removing the condition (0.10).

More precisely we can state the following result.

Corollary 0.2. *Let $n = 1$ and $p = 4$. Then under the same notations as in Theorem 0.2 we have that for $r < r_0$ the corresponding set \mathcal{B}_r is weakly orbitally stable. Indeed \mathcal{B}_r satisfies a straightened version of the property given in Definition 0.1, where we can replace $H_{rad}^1(\mathbb{R})$ by $H^1(\mathbb{R})$.*

We point out that the *weak orbital stability* stated in Theorem 0.2 under the condition (0.10), as well as in Corollary 0.2, is a byproduct of the general Cazenave-Lions strategy (see [6]), once the following compactness property (where we don't assume any radiality assumption) is established:

$$u_k \in S_r \cap B_1, \quad \mathcal{E}_s(u_k) \rightarrow \mathcal{J}_r \Rightarrow \exists x_k \in \mathbb{R}^n \text{ s.t.}$$

$$u_k(x + x_k) \text{ has a strong limit in } H^{1/2}(\mathbb{R}^n).$$

The main difficulty here being the fact that we have to deal with a local minimization problems (since the global minimization problem is meaningless, see (0.8)) and hence the application of the concentration-compactness argument is much more delicate. We also underline that if we look at the same minimization problems as above, under the extra radiality assumption (namely $u_k(x) = u_k(|x|)$),

then the compactness stated above occurs without any selection of the translation parameters x_k .

We would like to mention that at the best of our knowledge this is the first result about translation invariant equations, where stable solitary waves are proved to exist in the L^2 -supercritical regime.

In order to state our last result about existence/instability of ground states for HW, we need to introduce also the following functional:

$$\mathcal{P}(u) = \frac{1}{2} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 - \frac{n(p-1)}{2(p+1)} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1},$$

and the corresponding set:

$$(0.11) \quad \mathcal{M} = \{u \in H^{1/2}(\mathbb{R}^n) \text{ s.t. } \mathcal{P}(u) = 0\}.$$

It is well known (see [18]) that we have the following inclusion

$$\{w \in S_r \text{ s.t. } \mathcal{E}'_{hw}|_{S_r} = 0\} \subset \mathcal{M},$$

namely every critical point of the energy \mathcal{E}_{hw} on the constraint S_r belongs to the set \mathcal{M} . It is worth mentioning that this fact is reminiscent of the Pohozaev identity, which is here adapted to the case of HW. The following minimization problem will be crucial in the sequel:

$$\mathcal{I}_r = \inf_{S_r \cap \mathcal{M}} \mathcal{E}_{hw}(u).$$

Theorem 0.3. *Let $n \geq 1$ and $1 + \frac{2}{n} < p < 1 + \frac{2}{n-1}$. Then for every $r > 0$ we have:*

- $\mathcal{I}_r > -\infty$ and $\mathcal{A}_r \neq \emptyset$, where

$$\mathcal{A}_r := \{v \in S_r \cap \mathcal{M} \text{ s.t. } \mathcal{E}_{hw}(v) = \mathcal{I}_r\}.$$

Moreover any $v \in \mathcal{A}_r$ satisfies

$$\sqrt{-\Delta}v + \omega v - v|v|^{p-1} = 0$$

for a suitable $\omega \in \mathbb{R}$;

- assume $f(x) \in S_r \cap H_{rad}^1(\mathbb{R}^n)$ satisfies $\mathcal{E}_{hw}(f) < \mathcal{I}_r$ and $\mathcal{P}(f) < 0$, $n \geq 2$ and $u(t, x)$ is solution to (0.1) (where $A = \sqrt{-\Delta}$), then the following alternative holds: either the solution blows-up in finite time or $\|u(t, x)\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \geq e^{at}$ for suitable $a > 0$. In particular the set \mathcal{A}_r is not weakly orbitally stable for the flow associated with HW.

Notice that in the first part of the statement, which is mostly variational, we don't assume the radial symmetry. On the contrary in the statement about the evolution along the Cauchy problem we assume the radially. This is mainly due to the fact that at the best of our knowledge no global Cauchy theory is available without the radially assumption.

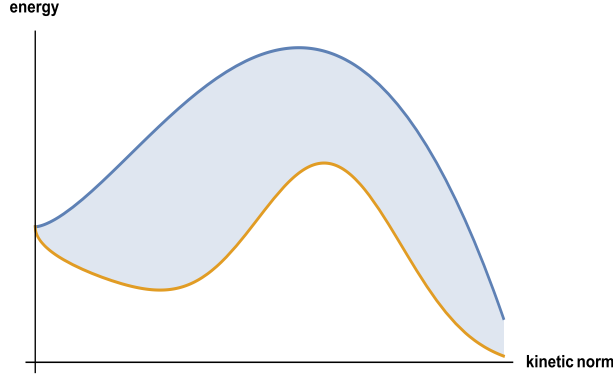


FIGURE 1. Qualitative behavior of the constrained energy functionals associated with sNLS and HW. In this qualitative picture the kinetic energy is given by $\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}$.

We also underline that our approach to prove the second part of Theorem 0.3, namely the norm inflation, is inspired by the work of Ogawa-Tsutsumi [12] that was based, in the context of the classical NLS, on the analysis of time derivative of the localized virial

$$M_{\varphi_R}(u) = 2 \operatorname{Im} \int \bar{u} \nabla \varphi_R \cdot \nabla u dx$$

where φ_R is a rescaled cut-off function such that $\nabla \varphi_R(x) \equiv x$ for $|x| \leq R$ and $\nabla \varphi_R(x) \equiv 0$ for $|x| \gg R$. This approach has been further extended by Boulenger-Himmelsbach-Lenzmann [3] in the non-local context with dispersion $(-\Delta)^s$ for $\frac{1}{2} < s < 1$. In this paper we shall take advantage of similar computations in the case of HW.

We point out that the discrepancy between the dynamics for sNLS and HW, revealed by Theorems 0.2 and 0.3 about the stability/instability of ground states (namely \mathcal{B}_r and \mathcal{A}_r) in the L^2 supercritical regime, is reminiscent of the results of [8, 9] for the dispersive equation describing a Boson Star:

$$i\partial_t u = \sqrt{m^2 - \Delta} u - \left(\frac{1}{|x|} \star |u|^2 \right) u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3.$$

Indeed, in [8] it has been proved that ground states are unstable by blow up if $m = 0$, while in [9] it is shown that the ground states are orbitally stable whenever $m > 0$. However our situation is rather different from the one describing a Boson Star, in fact in our case the constrained energy functional is always unbounded from below for any assigned L^2 constraint.

We conclude with a picture showing the main difference between the functionals \mathcal{E}_{hw} and \mathcal{E}_s revealed by Theorems 0.2 and 0.3, in the L^2 supercritical regime. In fact in the first case we have established the stability of the ground states, and

in the second case we have proved on the contrary its instability. For the HW (upper curve in the figure) the functional \mathcal{E}_{hw} admits a critical point of mountain pass type. For sNLS (lower curve in the figure) we have the existence of a local minimizer for \mathcal{E}_s .

1. THE CAUCHY THEORY FOR HW AND sNLS

The aim of this section is to prove a local/global existence and uniqueness result for the Cauchy problem (0.1). We need several tools that we shall exploit along the proof. We treat in some details the result for the HW, and we say at the end how to transfer the results at the level of sNLS. As usual we shall look for fixed point of the integral operator associated with the Cauchy problem for HW:

$$(1.1) \quad S_f(u) = e^{-it\sqrt{-\Delta}}f + i \int_0^t e^{-i(t-\tau)\sqrt{-\Delta}}u(\tau)|u(\tau)|^{p-1}d\tau.$$

where $f(x) \in H_{rad}^1(\mathbb{R}^n)$. We perform a fixed point argument in a suitable space $X_T \subset C([0, T]; H_{rad}^1(\mathbb{R}^n))$ that, as we shall see below, is provided by an interpolation between a Kato-smoothing type estimate and the usual energy estimates.

1.1. A Brezis-Gallouët-Strauss Type Inequality in $H_{rad}^{1/2+\epsilon}(\mathbb{R}^n)$. In this subsection we introduce two functional inequalities that will be useful respectively to achieve the local Cauchy theory and the globalization argument, following in the spirit the paper by Strauss (see [20]) and Brezis-Gallouët (see [5]).

Proposition 1.1. *For every $n \geq 2$ and $s > 1/2$ there exists a constant $C = C(s, n) > 0$ such that:*

$$(1.2) \quad \| |x|^{\frac{n-1}{2}} u \|_{L^\infty(\mathbb{R}^n)} \leq C \|u\|_{H^s(\mathbb{R}^n)};$$

$$(1.3) \quad \| |x|^{\frac{n-1}{2}} u \|_{L^\infty(\mathbb{R}^n)} \leq C \|u\|_{H^{1/2}(\mathbb{R}^n)} \sqrt{\ln \left(2 + \frac{\|u\|_{H^s(\mathbb{R}^n)}}{\|u\|_{H^{1/2}(\mathbb{R}^n)}} \right)},$$

for every $u \in H_{rad}^s(\mathbb{R}^n)$.

Proof. In the radial case it is well-known the Strauss estimate ([20])

$$|x|^{\frac{n-1}{2}} |u(x)| \leq C \|f\|_{H^1(\mathbb{R}^n)}, \quad \forall u \in H_{rad}^1(\mathbb{R}^n)$$

that has been extended in [19] to

$$(1.4) \quad |x|^{\frac{n-1}{2}} |u_j(x)| \leq C \|u_j\|_{H^{1/2}(\mathbb{R}^n)}, \quad \forall u \in H_{rad}^{1/2}(\mathbb{R}^n), \quad \forall j \geq 0.$$

Here we use the notation

$$u_j = \varphi_j \left(\sqrt{-\Delta} \right) u, \quad \forall j \geq 0,$$

and $\varphi_j(s)$ is the usual Paley-Littlewood decomposition, namely $\varphi_j(s) \in C_0^\infty((0, \infty))$ are non-negative function supported in $[2^{j-1}, 2^{j+1}]$, such that:

$$\sum_{j \geq 0} \varphi_j(s) = 1, \quad \forall s \geq 0.$$

Notice that the first estimate (1.2) follows by decomposing $u(x) = \sum_{j \geq 0} u_j(x)$ and by noticing that by Minkowski inequality and (1.4)

$$\| |x|^{\frac{n-1}{2}} u \|_{L^\infty(\mathbb{R}^n)} \leq \sum_j \| |x|^{\frac{n-1}{2}} u_j \|_{L^\infty(\mathbb{R}^n)} \leq C \sum_j \| u_j \|_{H^{1/2}(\mathbb{R}^n)} \leq C \| u \|_{H^s(\mathbb{R}^n)}$$

where at the last step we have used the Cauchy-Schwartz inequality and the assumption $s > 1/2$.

Concerning the proof of (1.3) we refine the argument above as follows:

$$\| |x|^{\frac{n-1}{2}} u \|_{L^\infty(\mathbb{R}^n)} \leq C \sum_{j=0}^{\infty} \| u_j \|_{H^{1/2}(\mathbb{R}^n)} = \underbrace{\sum_{j=0}^M \| u_j \|_{H^{1/2}(\mathbb{R}^n)}}_{S_1(M)} + \underbrace{\sum_{j=M+1}^{\infty} \| u_j \|_{H^{1/2}(\mathbb{R}^n)}}_{S_2(M)}$$

with M being sufficiently large integer. We can estimate these two terms by Cauchy-Schwartz as follows:

$$\begin{aligned} S_1(M) &\leq C \sqrt{M} \| u \|_{H^{1/2}(\mathbb{R}^n)}, \\ S_2(M) &\leq C 2^{-M(1/4+s/2)} \| u \|_{H^s(\mathbb{R}^n)} \end{aligned}$$

so we get

$$\| |x|^{\frac{n-1}{2}} u(x) \|_{\infty} \leq C \sqrt{M} \| u \|_{H^{1/2}(\mathbb{R}^n)} + C 2^{-M(1/4+s/2)} \| u \|_{H^s(\mathbb{R}^n)}.$$

We conclude by choosing:

$$M = \ln \left(2 + \frac{\| u \|_{H^s(\mathbb{R}^n)}}{\| u \|_{H^{1/2}(\mathbb{R}^n)}} \right).$$

□

1.2. Energy Estimates and Kato Smoothing. Next proposition is the key estimate for the linear propagator, that will suggest the space X_T where to perform a fixed point argument. In the sequel we shall use the notation

$$[x]_\delta = |x|^{1+\delta} + |x|^{1-\delta}.$$

Proposition 1.2. *Let $\delta > 0$ be fixed. We have the following bound*

$$\| [x]_\delta^{-\frac{1}{q}} e^{it\sqrt{-\Delta}} f \|_{L^q(\mathbb{R}; L^2(\mathbb{R}^n))} \leq C \| f \|_{L^2(\mathbb{R}^n)}$$

for every $q \in [2, \infty]$ and $C > 0$ is an universal constant that does not depend on q .

Proof. By interpolation it is sufficient to treat $q = 2, \infty$. The case $q = \infty$ is trivial and follows by the isometry $\|e^{it\sqrt{-\Delta}}f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$. The case $q = 2$ follows by combining the Kato smoothing (see [11], [15]) together with the following lemma that provides uniform weighted estimates for the resolvent associated with $\sqrt{-\Delta}$.

Lemma 1.1. *Let $\delta > 0$ be fixed, then we have the following uniform bounds:*

$$\|[x]_\delta^{-\frac{1}{2}}(\sqrt{-\Delta} - (\lambda + i\epsilon))^{-1}f\|_{L^2(\mathbb{R}^n)} \leq C\|[x]_\delta^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^n)}$$

where $C > 0$ does not depend on $\lambda, \epsilon > 0$.

Proof. We have the following identity

$$(\sqrt{-\Delta} - (\lambda + i\epsilon))^{-1} = (\sqrt{-\Delta} + \lambda + i\epsilon) \circ (-\Delta - (\lambda + i\epsilon)^2)^{-1}$$

and hence the desired estimate follows by the following well-known estimates available for the resolvent associated with the Laplacian operator $-\Delta$ (see [1], [16]):

$$\|[x]_\delta^{-\frac{1}{2}}\sqrt{-\Delta}(-\Delta - (\lambda + i\epsilon)^2)^{-1}f\|_{L^2(\mathbb{R}^n)} \leq C\|[x]_\delta^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^n)}$$

and

$$\|[x]_\delta^{-\frac{1}{2}}(-\Delta - (\lambda + i\epsilon)^2)^{-1}f\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{|\lambda + i\epsilon|}\|[x]_\delta^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^n)}.$$

□

The proof of Proposition 1.2 is complete.

□

Next we present a-priori estimates associated with the Duhamel operator.

Proposition 1.3. *Let $\delta > 0$ be fixed. For every $q_1 \in [2, \infty]$ and $q_2 \in (2, \infty]$ we get*

$$\|[x]_\delta^{-\frac{1}{q_1}} \int_0^t e^{i(t-s)\sqrt{-\Delta}} F(s) ds\|_{L^{q_1}(\mathbb{R}; L^2(\mathbb{R}^n))} \leq C\|[x]_\delta^{\frac{1}{q_2}} F\|_{L^{q_2'}(\mathbb{R}; L^2(\mathbb{R}^n))}.$$

Proof. The proof follows by combining Proposition 1.2 with the TT^* argument (see [10]) in conjunction with the Christ-Kiselev Lemma (see [7]).

More precisely let T be the following operator:

$$T : L^2(\mathbb{R}^n) \ni f \rightarrow e^{it\sqrt{-\Delta}}f \in X_q$$

where

$$\|G(t, x)\|_{X_q} = \|[x]_\delta^{-\frac{1}{q}} G(t, x)\|_{L^q(\mathbb{R}; L^2(\mathbb{R}^n))}.$$

Notice that T is continuous by Proposition 1.2, and hence by a duality argument we get the continuity of the operator

$$T^* : Y_q \ni F(t, x) \rightarrow \int_{\mathbb{R}} e^{-is\sqrt{-\Delta}} F(s) ds \in L^2(\mathbb{R}^n).$$

(here we have used the dual norm $\|G(t, x)\|_{Y_q} = \|[x]_\delta^{1/q} G(t, x)\|_{L^{q'}(\mathbb{R}; L^2(\mathbb{R}^n))}$.) As a consequence, by choosing respectively $q = q_1$ and $q = q_2$ in the estimates above, we deduce that the following operator is continuous:

$$T \circ T^* : Y_{q_2} \ni F(t, x) \rightarrow \int_{\mathbb{R}} e^{i(t-s)\sqrt{-\Delta}} F(s) ds \in X_{q_1}$$

and hence

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\sqrt{-\Delta}} F(s) ds \right\|_{X_{q_1}} \leq C \|F\|_{Y_{q_2}}.$$

In fact by a straightforward localization argument (namely choose $F(s, x)$ supported only for $s > 0$) we get

$$\left\| \int_0^\infty e^{i(t-s)\sqrt{-\Delta}} F(s) ds \right\|_{X_{q_1}} \leq C \|F\|_{Y_{q_2}}.$$

Notice that this estimate looks very much like the one that we want to prove, except that we would like to replace the integral \int_0^∞ by the truncated integral \int_0^t . This is possible thanks to the general Christ-Kiselev Lemma mentioned above, that works provided that $q'_2 < q_1$. □

1.3. Local Cauchy Theory in $H_{rad}^1(\mathbb{R}^d)$. We define the space X_T and we perform in X_T a contraction argument for the integral operator S_f (see (1.1)).

We introduce $q, \bar{q} > 2$ and $\delta > 0$ such that

$$(1.5) \quad -\frac{(n-1)(p-1)}{2} + \frac{1-\delta}{\bar{q}} = \frac{-1+\delta}{q}$$

where $u|u|^{p-1}$ is the nonlinearity.

Notice that q, \bar{q}, δ as above exist provided that $p \in (1, 1 + \frac{2}{n-1})$. Next we introduce the space X_T whose norm is defined as

$$(1.6) \quad \|u\|_{X_T} = \|u(t, x)\|_{L_T^\infty H^1(\mathbb{R}^n)} + \|[x]_\delta^{-\frac{1}{q}} \nabla_x u(t, x)\|_{L_T^q L^2(\mathbb{R}^n)} \\ + \|[x]_\delta^{-\frac{1}{\bar{q}}} u(t, x)\|_{L_T^{\bar{q}} L^2(\mathbb{R}^n)}$$

(here and below we use the notation $L_T^r(X) = L^r((0, T); X)$). Next we introduce a cut-off function $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi(x) = 0$ for $|x| > 2$ and $\psi(x) = 1$ for $|x| < 1$ and we write the forcing term $u|u|^{p-1} = \psi u|u|^{p-1} + (1-\psi)u|u|^{p-1}$. Then we have, by using Proposition 1.2 and Proposition 1.3 where we choose $(q_1, q_2) = (\infty, \bar{q})$ and $(q_1, q_2) = (q, \bar{q})$ and where we apply the operator ∇_x (that commutes with the equation):

$$\|S_f u\|_{X_T} \leq C \|f\|_{H^1(\mathbb{R}^n)} + C \|[x]_\delta^{\frac{1}{\bar{q}}} \nabla_x (\psi(u|u|^{p-1}))\|_{L_T^{\bar{q}'} L^2(\mathbb{R}^n)} \\ + C \|\nabla_x ((1-\psi)(u|u|^{p-1}))\|_{L_T^1 L^2(\mathbb{R}^n)}$$

where S_f is the integral equation defined in (1.1). Then we get by using the Leibnitz rule and the properties of the cut-off function ψ :

$$(1.7) \quad \|S_f u\|_{X_T} \leq C\|f\|_{H^1} + C\| |x|^{-\frac{(n-1)(p-1)}{2}} |x|^{\frac{1-\delta}{q}} (|x|^{\frac{n-1}{2}} |u|)^{p-1} \nabla_x u \|_{L_T^{\bar{q}'} L^2(|x|<2)} \\ + C\| |x|^{\frac{-1+\delta}{q}} (|x|^{\frac{n-1}{2}} |u|)^{p-1} \|_{L_T^{\bar{q}'} L^2(|x|<2)} + C\| |u|^{p-1} \nabla_x u \|_{L_T^1 L^2(|x|>1)}.$$

Next notice that by (1.5), we have the estimate $\| |x|^{\frac{n-1}{2}} u \|_{L^\infty(\mathbb{R}^d)} \leq C\|u\|_{H^1(\mathbb{R}^d)}$ for every $u \in H_{rad}^1(\mathbb{R}^d)$ and by Hölder in space and time we get:

$$\dots \leq C\|f\|_{H^1(\mathbb{R}^d)} + CT^{1-\frac{1}{q}-\frac{1}{q}} \|u\|_{L_T^\infty H^1(\mathbb{R}^d)}^{p-1} \|u\|_{X_T} + CT\|u\|_{L_T^\infty H^1(\mathbb{R}^d)}^p.$$

From this estimate one can conclude that

$$S_f : B_{X_T}(0, R) \rightarrow B_{X_T}(0, R)$$

for suitable $T, R > 0$ where

$$B_{X_T}(0, R) = \{v(t, x) \in X_T \text{ s.t. } \|v\|_{X_T} \leq R\}.$$

Next we endow the set $B_{X_T}(0, R)$ with the following distance:

$$d(u_1, u_2) = \|u_1 - u_2\|_{L_T^\infty L^2(\mathbb{R}^n)} + \|[x]_\delta^{-\frac{1}{q}} u(t, x)\|_{L_T^q L^2(\mathbb{R}^n)}.$$

It is easy to check that the metric space $(B_{X_T}(0, R), d)$ is complete. Then we conclude provided that we show that the map S_f is a contraction on this space. In order to do that we notice that by using the estimates in Proposition 1.3 (but we don't apply in this case the operator ∇_x) then we get:

$$\|S_f u_1(t) - S_f u_2(t)\|_{L_T^\infty L^2(\mathbb{R}^m)} + \|[x]_\delta^{-\frac{1}{q}} (S_f u_1 - S_f u_2)\|_{L_T^q L^2(\mathbb{R}^n)} \\ \leq CT^{1-\frac{1}{q}-\frac{1}{q}} (\|u_1\|_{X_T}^{p-1} + \|u_2\|_{X_T}^{p-1}) \|[x]_\delta^{-\frac{1}{q}} (u_1 - u_2)\|_{L_T^q L^2(\mathbb{R}^n)} \\ + CT(\|u_1\|_{X_T}^{p-1} + \|u_2\|_{X_T}^{p-1}) \|u_1 - u_2\|_{L_T^\infty L^2(\mathbb{R}^m)}$$

Hence by choosing $T > 0$ small enough and by recalling that $u_1, u_2 \in B_{X_T}(0, R)$ then we get:

$$d(S_f u_1, S_f u_2) \leq \frac{1}{2} d(u_1, u_2), \quad \forall u_1, u_2 \in B_{X_T}(0, R).$$

We conclude by using the contraction mapping principle.

1.4. Conditional Global Existence in $H_{rad}^1(\mathbb{R}^n)$ for $1 < p < 1 + \frac{2}{n}$.

First notice that for $1 < p < 1 + \frac{2}{n}$, then from Gagliardo-Nirenberg inequality we get

$$(1.8) \quad \sup_{(-T_-(f), T_+(f))} \|u(t, x)\|_{H^{1/2}(\mathbb{R}^n)} < \infty$$

where $(-T_-(f), T_+(f))$ is the maximal time of existence.

By arguing as in the subsection 1.3 (and by using the fact that $u(t, x)$ is a solution) we get:

$$\begin{aligned} \|u\|_{X_T} &\leq C\|f\|_{H^1(\mathbb{R}^n)} + C\| |x|^{-\frac{(n-1)(p-1)}{2}} |x|^{\frac{1-\delta}{q}} \nabla_x u(|x|^{\frac{n-1}{2}} |u|^{p-1}) \|_{L_T^{\bar{q}'} L^2(|x|<2)} \\ &\quad + C\| |u|^{p-1} \nabla_x u \|_{L_T^1 L^2(|x|>1)}. \end{aligned}$$

By (1.5) and Hölder in time we get:

$$\begin{aligned} \|u\|_{X_T} &\leq C\|f\|_{H^1(\mathbb{R}^n)} \\ &\quad + C\left(\| |x|^{\frac{1-\delta}{q}} \nabla_x u \|_{L_T^{\bar{q}'} L^2(|x|<2)} + T\|u\|_{L_T^\infty H^1(\mathbb{R}^n)}\right) \| |x|^{\frac{n-1}{2}} u \|_{L_T^\infty L^\infty(\mathbb{R}^n)}^{p-1} \\ &\leq C\|f\|_{H^1(\mathbb{R}^n)} + C(T^{1-\frac{1}{q}-\frac{1}{q}} \|u\|_{X_T} + T\|u\|_{L_T^\infty H^1(\mathbb{R}^n)}) \| |x|^{\frac{n-1}{2}} u \|_{L_T^\infty L^\infty(\mathbb{R}^n)}^{p-1}. \end{aligned}$$

Hence if we introduce the function $g(T) = \sup_{t \in (0, T)} \|u\|_{X_t}$ we obtain:

$$g(T) \leq C\|f\|_{H^1(\mathbb{R}^n)} + C \max\{T, T^{1-\frac{1}{q}-\frac{1}{q}}\} g(T) \| |x|^{\frac{n-1}{2}} u \|_{L_T^\infty L^\infty(\mathbb{R}^n)}^{p-1}.$$

By using (1.3) in conjunction with the assumption $\sup_t \|u(t, x)\|_{H^{1/2}(\mathbb{R}^n)} < \infty$, we have:

$$(1.9) \quad g(T) \leq C\|f\|_{H^1(\mathbb{R}^n)} + C \max\{T, T^{1-\frac{1}{q}-\frac{1}{q}}\} g(T) \ln^{\frac{p-1}{2}}(2 + Cg(T)).$$

Next we prove, as consequence of the estimate above, the following:

CLAIM Let $\bar{T} > 0$ be s. t. $C \max\{\bar{T}, \bar{T}^{1-\frac{1}{q}-\frac{1}{q}}\} \ln^{\frac{p-1}{2}}(2 + 2C^2\|f\|_{H^1(\mathbb{R}^n)}) = \frac{1}{2}$ then $g(\bar{T}) \leq 2C\|f\|_{H^1(\mathbb{R}^n)}$.

In order to prove the claim notice that if it is not true then there exists $\tilde{T} < \bar{T}$ such that $g(\tilde{T}) = 2C\|f\|_{H^1(\mathbb{R}^n)}$. Then by going back to the proof of (1.9) and by using the property $\tilde{T} < \bar{T}$, one can prove:

$$\begin{aligned} g(\tilde{T}) &\leq C\|f\|_{H^1(\mathbb{R}^n)} + C \max\{\tilde{T}, \tilde{T}^{1-\frac{1}{q}-\frac{1}{q}}\} g(\tilde{T}) \ln^{\frac{p-1}{2}}(2 + Cg(\tilde{T})) \\ &< C\|f\|_{H^1(\mathbb{R}^n)} + C \max\{\bar{T}, \bar{T}^{1-\frac{1}{q}-\frac{1}{q}}\} g(\tilde{T}) \ln^{\frac{p-1}{2}}(2 + 2C^2\|f\|_{H^1(\mathbb{R}^n)}) \\ &= C\|f\|_{H^1(\mathbb{R}^n)} + \frac{1}{2}g(\tilde{T}) \end{aligned}$$

and then we get $g(\tilde{T}) < 2C\|f\|_{H^1(\mathbb{R}^n)}$, hence contradicting the definition of \tilde{T} .

By an iteration argument (based on the claim above) we can construct a sequence \bar{T}_j such that

$$(1.10) \quad C \max\{\bar{T}_j, \bar{T}_j^{1-\frac{1}{q}-\frac{1}{q}}\} \ln^{\frac{p-1}{2}}(2 + 2C^2\|u(T_j)\|_{H^1(\mathbb{R}^n)}) = \frac{1}{2},$$

and

$$(1.11) \quad g(T_{j+1}) \leq 2^{j+1}C^{j+1}\|f\|_{H^1(\mathbb{R}^n)},$$

where $T_{j+1} = \bar{T}_1 + \dots + \bar{T}_j$. We claim that $T_j \rightarrow \infty$ as $j \rightarrow \infty$, and in this case we conclude. In fact if this is the case then the solution can be extended to the interval $[0, T_j]$ for every $j > 0$ and of course it implies global well-posedness since $T_j \rightarrow \infty$. Of course if there is a subsequence $T_{j_k} \geq 1$ then we conclude, and hence it is not restrictive to assume $T_j < 1$ at least for large j . In particular we get

$$(1.12) \quad \bar{T}_j^{1-\frac{1}{q}-\frac{1}{q}} = \max\{\bar{T}_j, \bar{T}_j^{1-\frac{1}{q}-\frac{1}{q}}\} \sim O((\ln \|u(T_j)\|_{H^1(\mathbb{R}^n)})^{-\frac{(p-1)}{2}}) \geq C(j^{-\frac{p-1}{2}})$$

where we used (1.10) and the fact that (1.11) implies $\|u(T_j)\|_{H^1(\mathbb{R}^n)} \leq 2^j C^j \|f\|_{H^1(\mathbb{R}^n)}$. We conclude since by (1.5) we can choose (q, \bar{q}) such that $\frac{1}{q} + \frac{1}{\bar{q}} = \frac{(n-1)(p-1)}{2} + \epsilon_0$ with $\epsilon_0 > 0$ arbitrarily small. In fact it implies, together with (1.12), the following estimate:

$$\bar{T}_{j_0+1} = \sum_{j=1}^{j_0} \bar{T}_j \geq C \sum_{j=1}^{j_0} j^{-\frac{p-1}{2-(n-1)(p-1)-2\epsilon_0}}$$

and the r.h.s. is divergent (for small $\epsilon_0 > 0$) provided that $\frac{p-1}{2-(n-1)(p-1)} < 1$, namely $p < 1 + \frac{2}{n}$.

1.5. Cauchy Theory for sNLS. The idea is to reduce the Cauchy theory for sNLS to the Cauchy theory for HW that has been established above. More precisely let us introduce the operator $L = \sqrt{1-\Delta} - \sqrt{-\Delta}$ and hence we can rewrite the Cauchy problem associated with sNLS as follows:

$$(1.13) \quad \begin{cases} i\partial_t u + \sqrt{-\Delta} = -Lu + u|u|^{p-1}, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = f(x) \in H_{rad}^1(\mathbb{R}^n). \end{cases}$$

Notice that the operator L corresponds in Fourier at the multiplier $\frac{1}{\sqrt{1+|\xi|^2+|\xi|}}$, and hence we have $L : H^s(\mathbb{R}^n) \rightarrow H^s(\mathbb{R}^n)$. Thanks to this property it is easy to check that we can perform a fixed point argument for (1.13) in the space X_T following the same argument used to solve above for HW. The minor change concerns the fact that the extra term Lu is absorbed in the nonlinear perturbation. Also the globalization argument given in the subsection 1.4 can be easily adapted to sNLS.

2. EXISTENCE AND STABILITY OF SOLITARY WAVES FOR SNLS

The main point along the proof of Theorem 0.2 is the proof of the compactness (up to translation) of the minimizing sequences associated with \mathcal{J}_r , as well as the proof of the fact that the minimizers belong to $B_{1/2} \cap S_r$, provided that r is small enough. This is sufficient in order to deduce that the minimizers are far away from the boundary and hence are constrained critical points. In particular they satisfy the Euler-Lagrange equation up to the Lagrange multiplier ω . Another delicate issue is to show that the local minimizers (namely the elements in \mathcal{B}_r according with the notation in Theorem 0.2) have indeed minimal energy between all the critical points of \mathcal{E}_s constrained on the whole sphere S_r , provided that $r > 0$ is small.

Next we shall focus on the points above, and we split the proofs in several steps. We also mention that the statement about the orbital stability it follows easily by the classical argument of Cazenave-Lions (see [6]) once a nice Cauchy theory has been established.

2.1. Local Minima Structure. We start with the following lemma that shows a local minima structure for the functional \mathcal{E}_s on the constraint S_r , for r small enough.

Proposition 2.1. *There exists $r_0 > 0$ such that:*

$$(2.1) \quad \inf_{\{u \in S_r \mid \|u\|_{H^{1/2}(\mathbb{R}^n)} = 1\}} \mathcal{E}_s(u) > \frac{1}{4}, \quad \forall r < r_0;$$

$$(2.2) \quad \inf_{\{u \in S_r \mid \|u\|_{H^{1/2}(\mathbb{R}^n)} \leq 2\sqrt{r}\}} \mathcal{E}_s(u) < \frac{r}{2}, \quad \forall r < r_0.$$

Proof. By the Gagliardo-Nirenberg inequality we get for some $\epsilon_0 > 0$:

$$(2.3) \quad \begin{aligned} \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} &\geq \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - C_0 r^{\gamma_0} \|u\|_{H^{1/2}(\mathbb{R}^n)}^{2+\epsilon_0} \\ &= \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 (1 - C_0 r^{\gamma_0} \|u\|_{H^{1/2}(\mathbb{R}^n)}^{\epsilon_0}), \quad \forall u \in S_r \end{aligned}$$

and hence

$$\frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} > \frac{1}{4}, \quad \forall u \in S_r, \quad \|u\|_{H^{1/2}(\mathbb{R}^n)} = 1.$$

Concerning the bound (2.2) notice that:

$$\begin{aligned} &\frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \\ &= \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{2} \|u\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|u\|_{L^2(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \end{aligned}$$

and by using Plancharel

$$\begin{aligned} \dots &= \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{2} \|u\|_{L^2(\mathbb{R}^n)}^2 + \frac{r}{2} - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \\ &= \frac{1}{2} \|u\|_H^2 + \frac{r}{2} - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}, \quad \forall u \in H^{1/2}(\mathbb{R}^n) \cap S_r \end{aligned}$$

where

$$\|u\|_H^2 = \int_{\mathbb{R}^n} \frac{|\xi|^2}{1 + \sqrt{1 + |\xi|^2}} |\hat{u}(\xi)|^2 d\xi.$$

In particular we get

$$(2.4) \quad \|u\|_H^2 \leq \frac{1}{2} \|u\|_{\dot{H}^1(\mathbb{R}^n)}^2, \quad \forall u \in \dot{H}^1(\mathbb{R}^n) \text{ s.t. } \hat{u}(\xi) = 0, \quad \forall |\xi| > 1.$$

Next we fix φ smooth, such that: $\hat{\varphi}(\xi) = 0, \forall |\xi| \geq 1$ and $\|\varphi\|_{L^2(\mathbb{R}^n)}^2 = r$. We also introduce $\varphi_\lambda(x)$ where $\hat{\varphi}_\lambda(\xi) = \lambda^{n/2} \hat{\varphi}(\lambda\xi)$, then we get by the inequalities above (next we restrict to $\lambda > 1$ in order to guarantee $\hat{\varphi}_\lambda(\xi) = 0, \forall |\xi| > 1$ and hence we can apply (2.4)):

$$\begin{aligned} &\frac{1}{2} \|\varphi_\lambda\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|\varphi_\lambda\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \\ &\leq \frac{1}{2} \|\varphi_\lambda\|_{\dot{H}^1(\mathbb{R}^n)}^2 + \frac{r}{2} - \frac{1}{p+1} \|\varphi_\lambda\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}. \end{aligned}$$

Notice that $\varphi_\lambda \in S_r$. Moreover by a rescaling argument we get

$$\frac{1}{2} \|\varphi_\lambda\|_{\dot{H}^1(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|\varphi_\lambda\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} < 0$$

for any λ large enough. We conclude since

$$\|\varphi_\lambda\|_{H^{1/2}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\hat{\varphi}|^2 \sqrt{1 + \frac{|\xi|^2}{\lambda^2}} d\xi \rightarrow \|\varphi\|_{L^2(\mathbb{R}^n)}^2 = r, \text{ as } \lambda \rightarrow \infty$$

and hence for λ large enough $\|\varphi_\lambda\|_{H^{1/2}(\mathbb{R}^n)} < 2\sqrt{r}$. □

2.2. Avoiding Vanishing. Next result will be crucial to exclude vanishing for the minimizing sequences. In the sequel $r_0 > 0$ is the number that appears in Proposition 2.1.

Proposition 2.2. *Assume $r < r_0$ and $u_k \in S_r \cap B_1$ be such that $\mathcal{E}_s(u_k) \rightarrow \mathcal{J}_r$ then $\liminf_{k \rightarrow \infty} \|u_k\|_{L^{p+1}(\mathbb{R}^n)} > 0$.*

Proof. Assume by the absurd that it is false. Then we get by Proposition 2.1

$$\frac{r}{2} > \mathcal{J}_r = \lim_{k \rightarrow \infty} \mathcal{E}_s(u_k) = \lim_{k \rightarrow \infty} \frac{1}{2} \|u_k\|_{H^{1/2}(\mathbb{R}^n)}^2.$$

This is a contradiction since $u_k \in S_r$ and hence $\|u_k\|_{H^{1/2}(\mathbb{R}^n)}^2 \geq r$.

□

2.3. Avoiding Dichotomy. Next result will be crucial to avoid dichotomy.

Proposition 2.3. *There exists $r_1 > 0$ such that for any $0 < r < l < r_1$ we have*

$$r\mathcal{J}_l < l\mathcal{J}_r.$$

We shall need the following result that allows us to get a bound on the size of the minimizing sequences.

Lemma 2.1. *There exists $r_2 > 0$ such that*

$$\inf_{\{u \in S_r \mid \|u\|_{H^{1/2}(\mathbb{R}^n)} \leq 2\sqrt{r}\}} \mathcal{E}_s(u) < \inf_{\{u \in S_r \mid \sqrt{r} \leq \|u\|_{H^{1/2}(\mathbb{R}^n)} < 1\}} \mathcal{E}_s(u), \quad \forall r < r_2.$$

Proof. In view of Proposition 2.1 it is sufficient to prove that

$$\inf_{\{u \in S_r \mid 2\sqrt{r} \leq \|u\|_{H^{1/2}(\mathbb{R}^n)} < 1\}} \left(\frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \right) > \frac{r}{2}.$$

In order to prove it, we go back to (2.3) and we get

$$\begin{aligned} & \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \\ & \geq \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^n)}^2 (1 - C_0 r^{\gamma_0} \|u\|_{H^{1/2}(\mathbb{R}^n)}^{\epsilon_0}), \quad \forall u \in S_r \end{aligned}$$

and hence

$$\dots \geq 2r(1 - C_0 r^{\gamma_0}), \quad \forall u \in S_r, \quad 2\sqrt{r} \leq \|u\|_{H^{1/2}(\mathbb{R}^n)} < 1.$$

We conclude provided that r is small enough.

□

We can now conclude the proof of Proposition 2.3. Fix $v_k \in S_r$, $\|v_k\|_{H^{1/2}(\mathbb{R}^n)} \leq 1$ such that $\lim_{k \rightarrow \infty} \mathcal{E}_s(v_k) = \mathcal{J}_r$.

Notice that by Lemma 2.1 we can assume $\|v_k\|_{H^{1/2}(\mathbb{R}^n)} < 2\sqrt{r}$. In particular we have

$$\sqrt{\frac{l}{r}} v_k \in S_l \text{ and } \sqrt{\frac{l}{r}} \|v_k\|_{H^{1/2}(\mathbb{R}^n)} < 2\sqrt{l},$$

and hence

$$\begin{aligned} \mathcal{J}_l & \leq \liminf_k \mathcal{E}_s\left(\sqrt{\frac{l}{r}} v_k\right) \\ & = \frac{1}{2} \frac{l}{r} \|v_k\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \frac{l^{\frac{p+1}{2}}}{r^{\frac{p+1}{2}}} \|v_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}. \end{aligned}$$

Recall that by Proposition 2.2 we can assume

$$\|v_k\|_{L^{p+1}(\mathbb{R}^n)} > \delta_0 > 0$$

and hence we can continue the estimate above as follows

$$\begin{aligned} \dots &= \frac{l}{r} \left(\frac{1}{2} \|v_k\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1} \|v_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \right) + \frac{1}{p+1} \left(\frac{l}{r} - \frac{l^{\frac{p+1}{2}}}{r^{\frac{p+1}{2}}} \right) \|v_k\|_{p+1}^{p+1} \\ &= \frac{l}{r} \mathcal{E}_s(v_k) + \left(\frac{l}{r} - \frac{l^{\frac{p+1}{2}}}{r^{\frac{p+1}{2}}} \right) \frac{\delta_0}{p+1} \leq \frac{l}{r} \mathcal{J}_r + \left(\frac{l}{r} - \frac{l^{\frac{p+1}{2}}}{r^{\frac{p+1}{2}}} \right) \frac{\delta_0}{p+1} < \mathcal{J}_r \frac{l}{r}. \end{aligned}$$

2.4. Conclusions. Notice that by Proposition 2.1 we deduce that the minimizers (if exist) have to belong necessarily to $B_{1/2}$. Next we prove the compactness, up to translations, of the minimizing sequences. Since now on we shall fix r small enough according with the Propositions above.

Let $u_k \in S_r$ be such that $\|u_k\|_{H^{1/2}(\mathbb{R}^n)} \leq 1$ and $\mathcal{E}_s(u_k) \rightarrow \mathcal{J}_r$, then by combining Proposition 2.2 and with the Lieb translation Lemma in $H^{1/2}(\mathbb{R}^n)$ (see [2]), we have that up to translation the weak limit of u_k is $\bar{u} \neq 0$. Our aim is to prove that \bar{u} is a strong limit in $L^2(\mathbb{R}^n)$. Hence if we denote $\|\bar{u}\|_{L^2(\mathbb{R}^n)}^2 = \bar{r}$ then it is sufficient to prove $\bar{r} = r$. Notice that we have by weak convergence

$$\|u_k - \bar{u}\|_{L^2(\mathbb{R}^n)}^2 + \|\bar{u}\|_{L^2(\mathbb{R}^n)}^2 = r + o(1)$$

and if we assume (by subsequence) $\|u_k - \bar{u}\|_{L^2(\mathbb{R}^n)}^2 \rightarrow t$ then we have $t + \bar{r} = r$. We shall prove that necessarily $t = 0$ and hence $r = \bar{r}$. Next by classical arguments, namely Brezis-Lieb Lemma (see [4]) and the Hilbert structure of $H^{1/2}(\mathbb{R}^n)$, we get

$$\mathcal{E}_s(u_k) = \mathcal{E}_s(u_k - \bar{u}) + \mathcal{E}_s(\bar{u}) + o(1) \geq \mathcal{J}_{\|u_k - \bar{u}\|_{L^2(\mathbb{R}^n)}^2} + \mathcal{J}_{\|\bar{u}\|_{L^2(\mathbb{R}^n)}^2} + o(1).$$

Passing to the limit as $k \rightarrow \infty$, and by recalling $t + \bar{r} = r$ we get

$$(2.5) \quad \mathcal{J}_{t+\bar{r}} = \mathcal{J}_r \geq \mathcal{J}_{\bar{r}} + \mathcal{J}_t.$$

On the other hand by Proposition 2.3 we get

$$(t + \bar{r}) \mathcal{J}_{\bar{r}} > \bar{r} \mathcal{J}_{t+\bar{r}} \text{ and } (t + \bar{r}) \mathcal{J}_t > t \mathcal{J}_{t+\bar{r}}$$

that imply $\mathcal{J}_{\bar{r}} + \mathcal{J}_t > \mathcal{J}_{t+\bar{r}}$ and it is in contradiction with (2.5).

As a last step we have to prove that the local minima (namely the elements in \mathcal{B}_r) minimize the energy \mathcal{E}_s among all the critical points of \mathcal{E}_s constrained to S_r , provided that $r > 0$ is small enough.

In order to prove this fact we shall prove the following property:

$$\exists r_0 > 0 \text{ s.t. } \forall r < r_0 \text{ the following occurs}$$

$$\|w\|_{H^{1/2}(\mathbb{R}^n)} < 1/2, \quad \forall w \in S_r \text{ s.t. } \mathcal{E}'_s|_{S_r} = 0, \mathcal{E}_s(w) < \mathcal{J}_r.$$

Once this fact is established then we can conclude easily since it implies that if $w \in S_r$ is a constraint critical points with energy below \mathcal{J}_r and $r < r_0$, then w can be used as test functions to estimate \mathcal{J}_r from above and we get $\mathcal{J}_r \leq \mathcal{E}_s(w)$, hence we have a contradiction.

In order to prove the property stated above recall that if $w \in S_r$ is a critical point of \mathcal{E}_s restricted to S_r , then notice that it has to satisfy the following Pohozaev type identity (this follows by an adaptation of the argument in [18] to sNLS):

$$\mathcal{Q}(w) = 0 \text{ where } \mathcal{Q}(w) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\xi|^2}{\sqrt{1+|\xi|^2}} |\hat{w}|^2 d\xi - \frac{n(p-1)}{2(p+1)} \|w\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}$$

and hence

$$\begin{aligned} \mathcal{E}_s(w) &= \mathcal{E}_s(w) - \frac{2}{n(p-1)} \mathcal{Q}(w) \\ &= \frac{1}{2} \|w\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{n(p-1)} \int_{\mathbb{R}^n} \frac{|\xi|^2}{\sqrt{1+|\xi|^2}} |\hat{w}|^2 d\xi \geq \frac{np-n-2}{2n(p-1)} \|w\|_{H^{1/2}(\mathbb{R}^n)}^2. \end{aligned}$$

Notice that $np-n-2 > 0$ if $p > 1 + \frac{2}{n}$. From the estimate above we get

$$\frac{np-n-2}{2n(p-1)} \|w\|_{H^{1/2}(\mathbb{R}^n)}^2 \leq \mathcal{E}_s(w) < \mathcal{J}_r < \frac{r}{2}$$

where we used Proposition 2.1 at the last step. It is now easy to conclude.

3. EXISTENCE/INSTABILITY OF SOLITARY WAVES FOR HW

This section is devoted to the proof of Theorem 0.3.

3.1. Existence of Minimizer. Even if Theorem 0.3 is stated for the energy $\mathcal{E}_{hw}(u)$ on S_r we shall work at the beginning on the unconstrained functional. At the end we shall come back to the constraint minimization problem as stated in Theorem 0.3.

The first result concerns the fact that the constraint \mathcal{M} (see (0.11)) is a natural constraint.

Lemma 3.1. *Let $v \in H^{1/2}(\mathbb{R}^n)$ be such that*

$$\mathcal{P}(v) = 0, \quad \mathcal{E}_{hw}(v) = \inf_{u \in \mathcal{M}} \mathcal{E}_{hw}(u)$$

then

$$\sqrt{-\Delta}v + v - v^p = 0$$

Proof. We notice that since v is minimizer then

$$\sqrt{-\Delta}v + v - v^p = \lambda \left(\sqrt{-\Delta}v - \frac{n(p-1)}{2} v^p \right)$$

We claim that $\lambda = 0$. Notice that by the equation above we get

$$(3.1) \quad \|v\|_{H^{1/2}(\mathbb{R}^n)}^2 + \|v\|_2^2 - \|v\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} = \lambda \|v\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{d\lambda(p-1)}{2} \|v\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}$$

Moreover since $\mathcal{P}(v) = 0$ we get

$$(3.2) \quad \|v\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 = \frac{n(p-1)}{(p+1)} \|v\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}.$$

Next notice that we have the following rescaling invariance

$$\mathcal{P}(u) = 0 \Rightarrow \mathcal{P}(\mu^{\frac{1}{p-1}} u(\mu x)) = 0$$

and hence since v is a minimizer we get $\frac{d}{d\mu} \mathcal{E}_{hw}(\mu^{\frac{1}{p-1}} v(\mu x))|_{\mu=1} = 0$ that by elementary computations gives

$$(3.3) \quad \begin{aligned} & \frac{1}{2} \left(\frac{p+1}{p-1} - n \right) \|v\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 + \frac{1}{2} \left(\frac{2}{p-1} - n \right) \|v\|_{L^2(\mathbb{R}^n)}^2 \\ & - \left(\frac{p+1}{p-1} - n \right) \frac{1}{p+1} \|v\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} = 0. \end{aligned}$$

By combining (3.1), (3.2), (3.3) we get easily $\lambda = 0$. □

The next result concerns the proof of the existence of a minimizer for the unconstrained problem.

Lemma 3.2. *Let $1 + \frac{2}{n} < p < 1 + \frac{2}{n-1}$. There exists $w \in H_{rad}^{1/2}(\mathbb{R}^n)$ such that*

$$\mathcal{P}(w) = 0, \quad \mathcal{E}_{hw}(w) = \inf_{u \in \mathcal{M}} \mathcal{E}_{hw}(u).$$

Proof. For simplicity we denote $\inf_{u \in \mathcal{M}} \mathcal{E}_{hw}(u) := \mathcal{E}_0$.

Notice that we have

$$(3.4) \quad \begin{aligned} \mathcal{E}_{hw}(u) &= \left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + \frac{1}{2} \|u\|_{L^2(\mathbb{R}^n)}^2, \\ &\forall u \in H^{1/2}(\mathbb{R}^n), \quad \mathcal{P}(u) = 0 \end{aligned}$$

and hence, since $\left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) > 0$ for $p > 1 + \frac{2}{n}$ we get $\mathcal{E}_0 \geq 0$. Let $w_k \in \mathcal{M}$ be a minimizing sequence.

We shall prove first the compactness of minimizing sequences by assuming radial symmetry, namely $w_k \in H_{rad}^{1/2}(\mathbb{R}^n)$ such that

$$(3.5) \quad \mathcal{E}_{hw}(w_k) \rightarrow \mathcal{E}_0, \quad w_k \in H_{rad}^{1/2}(\mathbb{R}^n), \quad \mathcal{P}(w_k) = 0.$$

In a second step we shall prove that it is not restrictive to assume that w_n can be assumed radially symmetric.

First of all notice that we get $\sup_n \|w_k\|_{H^{1/2}(\mathbb{R}^n)} < \infty$. In fact by using the constraint $\mathcal{P}(w_k) = 0$ it is sufficient to check that

$$(3.6) \quad \sup_k \|w_k\|_{L^2(\mathbb{R}^n)} + \|w_k\|_{L^{p+1}(\mathbb{R}^n)} < \infty$$

and it follows by the expression (3.4) of the energy on the constraint $\mathcal{P}(u) = 0$. The next step is to show that $\inf_k \|w_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} > 0$. It follows by the following chain of inequalities

$$\begin{aligned} \|w_k\|_{H^{1/2}(\mathbb{R}^n)}^2 &= \frac{n(p-1)}{p+1} \|w_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \\ &\leq C \|w_k\|_{L^2(\mathbb{R}^n)}^{\gamma_0} \|w_k\|_{L^{p+1}(\mathbb{R}^n)}^{2+\epsilon_0} \leq C \|w_k\|_{H^{1/2}(\mathbb{R}^n)}^{2+\epsilon_0} \end{aligned}$$

where we used the Gagliardo-Nirenberg inequality, the boundedness of $\|w_k\|_{L^2(\mathbb{R}^n)}$ (see (3.6)) and the Sobolev embedding $H^{1/2}(\mathbb{R}^n) \subset L^{p+1}(\mathbb{R}^n)$. As a consequence we get $\inf_k \|w_k\|_{H^{1/2}(\mathbb{R}^n)}^2 > 0$ and since $\mathcal{P}(w_k) = 0$ the same lower bound occurs for $\inf_k \|w_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} > 0$.

Next we introduce $\bar{w} \in H^{1/2}(\mathbb{R}^n)$ as the weak limit of w_k . We are done if we show that the convergence is strong. First of all notice that by the compactness of the Sobolev embedding $H_{rad}^{1/2}(\mathbb{R}^n) \rightarrow L^{p+1}(\mathbb{R}^n)$, we deduce $w_k \rightarrow \bar{w}$ in $L^{p+1}(\mathbb{R}^n)$ and hence $\bar{w} \neq 0$ since $\inf_n \|w_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} > 0$. Next notice that since $\mathcal{P}(w_k) = 0$ then

$$\frac{1}{2} \|\bar{w}\|_{H^{1/2}(\mathbb{R}^n)}^2 - \frac{n(p-1)}{2(p+1)} \|\bar{w}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \leq 0.$$

It implies by a continuity argument

$$\exists \bar{\lambda} \in (0, 1] \text{ s.t. } \mathcal{P}(\bar{\lambda} \bar{w}) = 0,$$

and in turn

$$\begin{aligned} \mathcal{E}_0 &\leq \mathcal{E}_{hw}(\bar{\lambda} \bar{w}) \\ &= \left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \bar{\lambda}^{p+1} \|\bar{w}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + \frac{1}{2} \bar{\lambda}^2 \|\bar{w}\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \bar{\lambda}^2 \left(\left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \|\bar{w}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + \frac{1}{2} \|\bar{w}\|_{L^2(\mathbb{R}^n)}^2 \right) \\ &\leq \left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \|\bar{w}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + \frac{1}{2} \|\bar{w}\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Notice that in the last inequality we get equality only in the case $\bar{\lambda} = 1$. Moreover by (3.4) and (3.5) we have

$$\left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \|\bar{w}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + \frac{1}{2} \|\bar{w}\|_{L^2(\mathbb{R}^n)}^2 \leq \mathcal{E}_0.$$

As a conclusion we deduce that above we have equality everywhere and hence the unique possibility is that $\bar{\lambda} = 1$ and we conclude.

Next we show via a Schwartz symmetrization argument that it is not restrictive to assume the minimizing sequence to be radially symmetric. Hence given $w_k \in H^{1/2}(\mathbb{R}^n)$ that satisfies (3.5) (but not necessarily radially symmetric), then we

can construct another radially symmetric sequence $u_k \in H_{rad}^{1/2}(\mathbb{R}^n)$ that satisfies (3.5). We introduce w_k^* as the Schwartz symmetrization of w_k . Notice that we have by standard facts about Schwartz symmetrization that

$$(3.7) \quad \liminf_{k \rightarrow \infty} \mathcal{E}_{hw}(w_k^*) \leq \mathcal{E}_0 \text{ and } \|w_k^*\|_{H^{1/2}(\mathbb{R}^n)}^2 \leq \frac{d(p-1)}{p+1} \|w_k^*\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}.$$

Notice that in principle $\mathcal{P}(w_k^*) \leq 0$. On the other hands

$$\exists \lambda_k \in (0, 1] \text{ s.t. } \mathcal{P}(\lambda_k w_k^*) = 0.$$

We conclude if we show that $\lambda_k \rightarrow 1$. In fact in this case it is easy to check that $\lambda_k w_k^* \in H_{rad}^{1/2}(\mathbb{R}^n)$, $\lambda_k w_k^* \in \mathcal{M}$ and $\mathcal{E}_{hw}(\lambda_k w_k^*) - \mathcal{E}_{hw}(w_k^*) \rightarrow 0$ and hence we conclude by (3.7) that $\mathcal{E}_{hw}(\lambda_k w_k^*) \rightarrow \mathcal{E}_0$.

In order to prove $\lambda_k \rightarrow 1$ we notice that by (3.4) we have

$$\begin{aligned} \mathcal{E}_0 \leq \mathcal{E}_{hw}(\lambda_k w_k^*) &= \left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \lambda_k^{p+1} \|w_k^*\|_{p+1}^{p+1} + \frac{1}{2} \lambda_k^2 \|w_k^*\|_2^2 \\ &\leq \left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \lambda_k^2 \|w_k^*\|_{p+1}^{p+1} + \frac{1}{2} \lambda_k^2 \|w_k\|_{L^2(\mathbb{R}^n)}^2 \\ &= \left(\frac{n(p-1)}{2(p+1)} - \frac{1}{p+1} \right) \lambda_k^2 \|w_k\|_{p+1}^{p+1} + \frac{1}{2} \lambda_k^2 \|w_k\|_{L^2(\mathbb{R}^n)}^2 = \lambda_k^2 \mathcal{E}_{hw}(w_k) \end{aligned}$$

where we used (3.4) at the last step. We deduce that $\lambda_k \rightarrow 1$ since $\mathcal{E}_{hw}(w_k) \rightarrow \mathcal{E}_0$. \square

We can now deduce for every $r > 0$ the existence of solitary waves belonging to S_r that moreover are minimizers of \mathcal{E}_{hw} constraint to $S_r \cap \mathcal{M}$. In fact let w be as in Lemma 3.2. Notice that by Lemma 3.1 we get

$$\sqrt{-\Delta} w + w - w^p = 0.$$

Moreover it is clear that $w \in \mathcal{A}_{r_0}$ where $r_0 = \|w\|_{L^2(\mathbb{R}^n)}^2$. Hence the first part of Theorem 0.3 is proved for $r = r_0$. The case of a generic r can be achieved by a straightforward rescaling argument.

3.2. Inflation of $H^{1/2}$ -norm for $\mathcal{P}(f) < 0$ and $\mathcal{E}_{hw}(f) < \mathcal{I}_r$. In this section we follow the approach of [3]. In the sequel the radially symmetric function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as

$$(3.8) \quad \varphi(r) = \begin{cases} \frac{r^2}{2} & \text{for } r \leq 1; \\ \text{const} & \text{for } r \geq 10. \end{cases}$$

with $\varphi''(t) \leq 2$ for $r > 0$, and we introduce the rescaled function $\varphi_R : \mathbb{R}^n \rightarrow \mathbb{R}$ as $\varphi_R(x) := R^2 \varphi(\frac{x}{R})$.

We define the localized virial in the spirit of Ogawa-Tsutsumi [12]

$$(3.9) \quad M_\varphi(u) = 2 \operatorname{Im} \int_{\mathbb{R}^n} \bar{u} \nabla \varphi \cdot \nabla u dx$$

In Lemma A.1 of [3] it is shown that $M_\varphi(u)$ can be bounded, as follows:

$$(3.10) \quad |M_\varphi(u)| \leq C \left(\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \right)$$

where the constant C depends only on $\|\nabla\varphi\|_{W^{1,\infty}(\mathbb{R}^n)}$ and on the space dimension. The following Lemma is crucial for our result.

Lemma 3.3 (Lemma 2.1, [3]). *Let $n \geq 2$, for any $f \in H_{rad}^1(\mathbb{R}^n)$ we have:*

$$(3.11) \quad \frac{d}{dt} M_\varphi(u) = \int_0^\infty m^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} 4\partial_k \bar{u}_m (\partial_{lk}^2 \varphi) \partial_l u_m - (\Delta^2 \varphi) |u_m|^2 dx \right) dm - \frac{2(p-1)}{p+1} \int_{\mathbb{R}^n} (\Delta \varphi) |u|^{p+1} dx$$

where $u_m(t, x) := \sqrt{\frac{1}{\pi}} \mathcal{F}^{-1} \left(\frac{\hat{u}(t, \xi)}{\xi^{2+m^2}} \right)$ and $u(t, x)$ is the unique solution to HW with initial condition $u(0, x) = f(x)$.

In the sequel we use the following Stein–Weiss inequality for radially symmetric functions due to Rubin [17] in general space dimension n .

Theorem 3.1 (Rubin [17]). *Let $n \geq 2$ and $0 < s < n$. Then for all $u \in \dot{H}_{rad}^s(\mathbb{R}^n)$ we have:*

$$(3.12) \quad \left(\int_{\mathbb{R}^n} |u(x)|^r |x|^{-\beta r} dx \right)^{1/r} \leq C(n, s, r, \beta) \|u\|_{\dot{H}^s(\mathbb{R}^n)},$$

where $r \geq 2$ and

$$(3.13) \quad -(n-1) \left(\frac{1}{2} - \frac{1}{r} \right) \leq \beta < \frac{n}{r},$$

$$(3.14) \quad \frac{1}{r} = \frac{1}{2} + \frac{\beta - s}{n}.$$

As a special case of Rubin theorem when the dimension is $n \geq 2$, $r = p+1$, $s = \frac{1}{4}$, $\beta = \frac{p(-2n+1)+2n+1}{4(p+1)}$ we have the following inequality

$$(3.15) \quad \left(\int_{\mathbb{R}^n} |u(x)|^{p+1} |x|^{-\frac{p(-2n+1)+2n+1}{4}} dx \right)^{\frac{1}{p+1}} \leq C \|u\|_{\dot{H}^{1/4}(\mathbb{R}^n)}$$

that holds if $1 < p \leq 3$, which is satisfied since we are assuming $1 + \frac{2}{n} < p < 1 + \frac{2}{n-1}$. As a byproduct of (3.15) and by noticing that $\frac{p(-2n+1)+2n+1}{4} < 0$ (this follows by the fact $p > 1 + \frac{2}{n}$) we have the following crucial decay:

$$(3.16) \quad \left(\int_{|x| \geq R} |u(x)|^{p+1} dx \right) \leq C R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{\dot{H}^{1/4}(\mathbb{R}^n)}^{p+1} \\ \leq C R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{L^2(\mathbb{R}^n)}^{\frac{p+1}{2}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^{\frac{p+1}{2}}.$$

We shall also need the following result.

Lemma 3.4. *For every $\delta, r > 0$ we have*

$$\sup_{\{u \in S_r | \mathcal{P}(u) < 0, \mathcal{E}_{hw}(u) \leq \mathcal{I}_r - \delta\}} \mathcal{P}(u) < 0.$$

Proof. Assume by the absurd that it is false, then for some $\delta_0, r_0 > 0$, we get

$$\exists u_k \in S_{r_0} \text{ s.t. } \mathcal{E}_{hw}(u_k) \leq \mathcal{I}_{r_0} - \delta_0, \mathcal{P}(u_k) < 0, \mathcal{P}(u_k) \rightarrow 0.$$

As a first remark we get

$$(3.17) \quad \sup_k \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)} < \infty.$$

In fact it follows by

$$\frac{np - n - 2}{2n(p - 1)} \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 = \mathcal{E}_{hw}(u_k) - \frac{2}{n(p - 1)} \mathcal{P}(u_k)$$

and we conclude since \limsup of the r.h.s. is below $\mathcal{I}_{r_0} - \delta_0$. Moreover we have

$$(3.18) \quad \inf_k \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)} > 0.$$

In fact notice that by assumption

$$(3.19) \quad \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 = \frac{n(p - 1)}{p + 1} \|u_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + \mathcal{P}(u_k)$$

where $\mathcal{P}(u_k) \rightarrow 0, \mathcal{P}(u_k) < 0$.

By combining this fact with the Gagliardo-Nirenberg inequality and by recalling that $\|u_k\|_{L^2(\mathbb{R}^n)}^2 = r_0 > 0$ we get for suitable universal constants $C_0, \epsilon_0 > 0$ (that depend from p, r_0)

$$\|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 \leq C_0 \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^{2+\epsilon_0},$$

and it implies (3.18). Notice also that by a simple continuity argument

$$(3.20) \quad \exists \lambda_k \in (0, 1) \text{ s.t. } \mathcal{P}(\lambda_k u_k) = 0.$$

We claim that $\lambda_k \rightarrow 1$. In fact we get by definition of \mathcal{P} we get:

$$(3.21) \quad \lambda_k^2 \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 = \frac{n(p - 1)}{p + 1} \lambda_k^{p+1} \|u_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}.$$

By combining the identities (3.21) and (3.19) above we get

$$\frac{n(p - 1)}{p + 1} (\lambda_k^{p-1} - 1) \|u_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} = \mathcal{P}(u_k) \rightarrow 0.$$

We conclude that $\lambda_k \rightarrow 1$ provided that we show $\inf_k \|u_k\|_{L^{p+1}(\mathbb{R}^n)} > 0$. Of course it is true otherwise we get

$$(3.22) \quad 0 = \lim_{k \rightarrow \infty} \mathcal{P}(u_k) = \lim_{k \rightarrow \infty} \frac{1}{2} \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2$$

which is in contradiction with (3.18). As a consequence of the fact $\lambda_k \rightarrow 1$ we deduce $\|\lambda_k u_k\|_2^2 = r_k \rightarrow r_0$ and hence by (3.20) we get $\mathcal{E}_{hw}(\lambda_k u_k) \geq \mathcal{I}_{r_k} \rightarrow \mathcal{I}_{r_0}$ (the

last limit follows by elementary considerations). In particular we get $\mathcal{E}_{hw}(\lambda_k u_k) \geq \mathcal{I}_{r_0} - \frac{\delta_0}{2}$. Moreover

$$\begin{aligned} & \mathcal{E}_{hw}(u_k) - \mathcal{E}_{hw}(\lambda_k u_k) \\ &= \frac{1}{2}(1 - \lambda_k^2) \|u_k\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 - \frac{1}{p+1}(1 - \lambda_k^{p+1}) \|u_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \rightarrow 0 \end{aligned}$$

where we used (3.17) with $\lambda_k \rightarrow 1$. We get a contradiction since $\mathcal{E}_{hw}(\lambda_k u_k) \geq \mathcal{I}_{r_0} - \frac{\delta_0}{2}$ and $\mathcal{E}_{hw}(u_k) \leq \mathcal{I}_{r_0} - \delta_0$. \square

Lemma 3.5 (Localized virial identity for HW). *There exists a constant $C > 0$ such that*

$$(3.23) \quad \frac{d}{dt} M_{\varphi_R}(u) \leq 4\mathcal{P}(u) + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{L^2(\mathbb{R}^n)}^{\frac{p+1}{2}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^{\frac{p+1}{2}})$$

for any $u(t, x)$ radially symmetric solution to HW.

Proof. In [3] it is shown (by choosing $s = \frac{1}{2}$) the following estimates:

$$4 \int_0^\infty m^{1/2} \int_{\mathbb{R}^n} \partial_k \bar{u}_m (\partial_{lk} \varphi_R) \partial_l u_m dx dm \leq 2 \|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2$$

and

$$\left| \int_0^\infty m^{1/2} \int_{\mathbb{R}^n} (\Delta^2 \varphi_R) |u_m|^2 dx dm \right| \leq CR^{-1}.$$

Concerning the last term in (3.11), namely $-\frac{2(p-1)}{p+1} \int_{\mathbb{R}^n} (\Delta \varphi_R) |u|^{p+1} dx$, we have

$$\begin{aligned} & -\frac{2(p-1)}{p+1} \int_{\mathbb{R}^n} (\Delta \varphi_R) |u|^{p+1} dx \\ &= -\frac{2n(p-1)}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx - \frac{2(p-1)}{p+1} \int_{\mathbb{R}^n} (\Delta \varphi_R - n) |u|^{p+1} dx. \end{aligned}$$

Notice that $\Delta \varphi_R = n$ on $\{|x| \leq R\}$, hence by recalling (3.16) and summarizing the estimates above we get:

$$\begin{aligned} \frac{d}{dt} M_{\varphi_R}(u) &\leq 2 \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 - \frac{2n(p-1)}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx \\ &\quad + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{L^2(\mathbb{R}^n)}^{\frac{p+1}{2}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^{\frac{p+1}{2}}) \end{aligned}$$

which is equivalent to

$$\frac{d}{dt} M_{\varphi_R}(u) \leq 4\mathcal{P}(u) + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{L^2(\mathbb{R}^n)}^{\frac{p+1}{2}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^{\frac{p+1}{2}}).$$

\square

We can now conclude the proof on the inflation of the norms, in the case that the solution exists globally in time. First notice that by Lemma 3.4 we get $\mathcal{P}(u(t, x)) < -\delta < 0$, and we claim that it implies

$$(3.24) \quad \inf_t \|u(t, x)\|_{\dot{H}^{1/2}(\mathbb{R}^n)} > 0$$

and for R sufficiently large

$$(3.25) \quad \frac{d}{dt} M_{\varphi_R}(u) \leq 2\mathcal{P}(u) - \alpha \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 < -\alpha \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2.$$

for some constant $\alpha > 0$. Let us first prove (3.24) and assume by contradiction the existence of a sequence of times t_n such that $\lim_{n \rightarrow \infty} \|u(t_n, x)\|_{\dot{H}^{1/2}}^2 = 0$. This fact implies by Sobolev embedding and conservation of the mass, that $\mathcal{P}(u(t_n, x)) \rightarrow 0$ which contradicts $\mathcal{P}(u(t, x)) < -\delta < 0$.

Now let us prove (3.25). By using (3.23) it is sufficient to prove that there exists α sufficiently small such that

$$4\mathcal{P}(u) + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{L^2(\mathbb{R}^n)}^{\frac{p+1}{2}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^{\frac{p+1}{2}}) \leq 2\mathcal{P}(u) - \alpha \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2$$

Notice that $\frac{p+1}{2} < 2$ and thanks to (3.24) and conservation of the mass we have the inequality

$$\begin{aligned} 4\mathcal{P}(u) + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{L^2(\mathbb{R}^n)}^{\frac{p+1}{2}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^{\frac{p+1}{2}}) \\ \leq 4\mathcal{P}(u) + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2) \end{aligned}$$

and hence it suffices to show that

$$(3.26) \quad 2\mathcal{P}(u) + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2) + \alpha \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 < 0$$

to get (3.25).

From the identity

$$\mathcal{E}_{hw}(u) - \frac{2}{n(p-1)} \mathcal{P}(u) = \frac{r}{2} + \frac{n(p-1)-2}{2n(p-1)} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2$$

we get

$$\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 \leq \frac{2n(p-1)}{n(p-1)-2} \mathcal{E}_{hw}(u) - \frac{4}{n(p-1)-2} \mathcal{P}(u).$$

As a consequence we get

$$\begin{aligned} 2\mathcal{P}(u) + C(R^{-1} + R^{\frac{p(-2n+1)+2n+1}{4}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2) + \alpha \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 \\ < \left(2 - \frac{4C}{n(p-1)-2} R^{\frac{p(-2n+1)+2n+1}{4}} - \frac{4\alpha}{n(p-1)-2}\right) \mathcal{P}(u) \\ + \frac{2nC(p-1)}{n(p-1)-2} R^{\frac{p(-2n+1)+2n+1}{4}} \mathcal{E}_{hw}(f) + \frac{2n\alpha(p-1)}{n(p-1)-2} \mathcal{E}_{hw}(f) + CR^{-1} \end{aligned}$$

Notice that we can conclude (3.26) since $\mathcal{P}(u(t, x)) < -\delta$ and hence it is sufficient to select α very small and R very large. By combining (3.24) with (3.10) we get

$$(3.27) \quad |M_{\varphi_R}(u)| \leq C(R) \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2.$$

From (3.25) we deduce that we can select $t_1 \in \mathbb{R}$ such that $M_{\varphi_R}(u(t)) \leq 0$ for $t \geq t_1$ and $M_{\varphi_R}(u(t_1)) = 0$. Hence integrating (3.25) we obtain

$$M_{\varphi_R}(u(t)) \leq -\alpha \int_{t_1}^t \|u(s)\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 ds.$$

Now by using (3.27) we get the integral inequality

$$|M_{\varphi_R}(u(t))| \geq C(R, \alpha) \int_{t_1}^t |M_{\varphi_R}(u(s))| ds,$$

which yields an exponential lower bound.

3.3. Instability of \mathcal{A}_r . Given $v(x) \in \mathcal{A}_r$ we shall show that there exists a sequence $\lambda_k \rightarrow 1$, $\lambda_k > 1$ such that

$$\mathcal{P}(\lambda_k^{d/2} v(\lambda_k x)) < 0, \quad \mathcal{E}_{hw}(\lambda_k^{d/2} v(\lambda_k x)) < \mathcal{E}_{hw}(v) = \mathcal{I}_r.$$

Then by denoting with $v_k(t, x)$ the unique solution to HW such that $v_k(0, x) = \lambda_k^{d/2} v(\lambda_k x)$ we get, by the inflation of norm proved in the previous subsection, that $v_k(t, x)$ are unbounded in $H^{1/2}$ for large time, despite to the fact that they are arbitrary close to $v(x)$ at the initial time $t = 0$. Of course it implies the instability of \mathcal{A}_r .

In order to prove the existence of λ_k as above, we introduce the functions

$$h : (0, \infty) \ni \lambda \rightarrow \mathcal{P}(\lambda^{d/2} v(\lambda x)) = \frac{1}{2} \lambda \|v\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 - \frac{n(p-1)}{2(p+1)} \lambda^{n(p-1)/2} \|v\|_{L^{p+1}(\mathbb{R}^n)}^{p+1},$$

$$\begin{aligned} g : (0, \infty) \ni \lambda &\rightarrow \mathcal{E}_{hw}(\lambda^{n/2} v(\lambda x)) \\ &= \frac{1}{2} \lambda \|v\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 + \|v\|_{L^2(\mathbb{R}^n)}^2 - \frac{1}{p+1} \lambda^{n(p-1)/2} \|v\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}. \end{aligned}$$

Notice that since $v(x) \in \mathcal{M}$ we get $h(1) = 0$ and hence by elementary analysis of the function h we deduce that $h(\lambda) < 0$ for every $\lambda > 1$. Moreover again by the fact that $v(x) \in \mathcal{M}$ we get $g'(1) = 0$ and since $g(1) = \mathcal{E}_{hw}(v) = \mathcal{I}_r$ we deduce that $g(\lambda) < \mathcal{I}_r$ for every $\lambda > 0$. It is now easy to conclude the existence of λ_k with the desired property.

4. APPENDIX

In this appendix we prove a global existence result for 1-d quartic sNLS with initial condition in $H^{3/2}(\mathbb{R})$. It is worth mentioning that the same argument used in [13], where it is treated 1-d quartic HW, can be adapted to sNLS. Hence the result stated below can be improved by assuming $f(x) \in H^1(\mathbb{R})$. However we want to give the argument below, since we believe that it is more transparent and slightly simpler. In particular it does not involve the use of fractional Leibnitz rules as in [13]. Moreover, in our opinion, it makes more clear the argument that stands behind the modified energy technique, which is a basic tool in [13] and that hopefully will be a basic tool to deal with other situations (see for instance [14]).

Theorem 4.1. *Let us fix $n = 1$ and $p = 4$. Assume that $u(t, x)$ solves sNLS with initial datum $f(x) \in H^{3/2}(\mathbb{R})$ and assume moreover that*

$$\sup_{(-T_-(f), T_+(f))} \|u(t, x)\|_{H^{1/2}(\mathbb{R})} < \infty,$$

where $(-T_-(f), T_+(f))$ is the maximal interval of existence. Then necessarily $T_{\pm}(f) = \infty$, namely the solution is global.

Proof. We have to show that the norm $\|u(t, x)\|_{H^{3/2}}$ cannot blow-up in finite time. In order to do that we notice that

$$i\partial_t(\partial_x u) = (\sqrt{1 - \Delta})\partial_x u - \partial_x(u|u|^3)$$

and hence if we multiply this equation by $\partial_t(\partial_x \bar{u})$, we integrate by parts and we get the real part, then we obtain:

$$\begin{aligned} 0 &= \Re \int_{\mathbb{R}} (\sqrt{1 - \Delta})\partial_x u \partial_t(\partial_x \bar{u}) dx - \Re \int_{\mathbb{R}} \partial_t(\partial_x \bar{u}) \partial_x(u|u|^3) dx \\ &= \Re \int_{\mathbb{R}} (1 - \Delta)^{1/4} \partial_x u \partial_t(\partial_x (1 - \Delta)^{1/4} \bar{u}) dx - \Re \int_{\mathbb{R}} \partial_t(\partial_x \bar{u}) \partial_x u |u|^3 dx - \Re \int_{\mathbb{R}} \partial_t(\partial_x \bar{u}) u \partial_x |u|^3 dx \end{aligned}$$

and hence

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{dt} \|(1 - \Delta)^{1/4} \partial_x u\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \int_{\mathbb{R}} \partial_t(|\partial_x u|^2) |u|^3 dx - \frac{3}{2} \Re \int_{\mathbb{R}} \partial_t(\partial_x \bar{u}) u \partial_x |u|^2 |u| dx \\ &= \frac{1}{2} \frac{d}{dt} \|(1 - \Delta)^{1/4} \partial_x u\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\partial_x u|^2 |u|^3 dx + \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 \partial_t(|u|^3) dx \\ &\quad - \frac{3}{2} \Re \int_{\mathbb{R}} \partial_t(\partial_x \bar{u}) u \partial_x \bar{u} |u| dx - \frac{3}{2} \Re \int_{\mathbb{R}} \partial_t(\partial_x \bar{u}) \partial_x u |u|^3 dx. \end{aligned}$$

We can continue as follows

$$0 = \frac{1}{2} \frac{d}{dt} \|(1 - \Delta)^{1/4} \partial_x u\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\partial_x u|^2 |u|^3 dx + \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 \partial_t(|u|^3) dx$$

$$\begin{aligned}
& -\frac{3}{4}\Re \int_{\mathbb{R}} \partial_t [(\partial_x \bar{u})]^2 u^2 |u| dx - \frac{3}{4}\Re \int_{\mathbb{R}} \partial_t (|\partial_x u|^2) |u|^3 dx \\
& = \frac{1}{2} \frac{d}{dt} \|(1 - \Delta)^{1/4} \partial_x u\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\partial_x u|^2 |u|^3 dx + \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 \partial_t (|u|^3) dx \\
& \quad - \frac{3}{4} \frac{d}{dt} \Re \int_{\mathbb{R}} (\partial_x \bar{u})^2 u^2 |u| dx + \frac{3}{4} \Re \int_{\mathbb{R}} (\partial_x \bar{u})^2 \partial_t (u^2 |u|) dx \\
& \quad - \frac{3}{4} \frac{d}{dt} \Re \int_{\mathbb{R}} |\partial_x u|^2 |u|^3 dx + \frac{3}{4} \Re \int_{\mathbb{R}} |\partial_x u|^2 \partial_t (|u|^3) dx.
\end{aligned}$$

Summarizing we get

$$(4.1) \quad \frac{d}{dt} \mathcal{E}(u) = -\frac{5}{4} \int_{\mathbb{R}} |\partial_x u|^2 \partial_t (|u|^3) dx - \frac{3}{4} \Re \int_{\mathbb{R}} (\partial_x \bar{u})^2 \partial_t (u^2 |u|) dx.$$

where

$$\mathcal{E}(u) = \frac{1}{2} \|(1 - \Delta)^{1/4} \partial_x u\|_{L^2(\mathbb{R})}^2 - \frac{5}{4} \int_{\mathbb{R}} |\partial_x u|^2 |u|^3 dx - \frac{3}{4} \Re \int_{\mathbb{R}} (\partial_x \bar{u})^2 u^2 |u| dx.$$

Notice that since $u(t, x)$ solves sNLS we deduce that the r.h.s. in (4.1) can be estimated by the following quantity (up to a constant):

$$\begin{aligned}
& \int_{\mathbb{R}} |\partial_x u|^2 |\sqrt{1 - \Delta} u| |u|^2 dx + \int_{\mathbb{R}} |\partial_x u|^2 |u|^6 dx \\
& \leq \|u\|_{W^{1,4}(\mathbb{R})}^2 \|u\|_{H^1(\mathbb{R})} \|u\|_{L^\infty(\mathbb{R})}^2 + \|u\|_{H^1(\mathbb{R})}^2 \|u\|_{L^\infty(\mathbb{R})}^6.
\end{aligned}$$

Next by using the Gagliardo-Nirenberg inequality $\|u\|_{W^{1,4}(\mathbb{R})}^2 \leq C \|u\|_{H^{3/2}(\mathbb{R})}^2 \|u\|_{H^1(\mathbb{R})}$, and the Brezis-Gallouët inequality (see [5]), together with the assumption on the boundedness of $H^{1/2}$ -norm of $u(t, x)$, we can continue the estimate above as follows:

$$\begin{aligned}
(4.2) \quad \text{r.h.s. (4.1)} & \leq C \|u\|_{H^{3/2}(\mathbb{R})} \|u\|_{H^1(\mathbb{R})}^2 \ln(2 + \|u\|_{H^{3/2}(\mathbb{R})}) \\
& \quad + \|u\|_{H^{3/2}(\mathbb{R})} \ln^3(2 + \|u\|_{H^{3/2}(\mathbb{R})}) \\
& \leq C \|u\|_{H^{3/2}(\mathbb{R})}^2 \ln(2 + \|u\|_{H^{3/2}(\mathbb{R})}) + \|u\|_{H^{3/2}(\mathbb{R})} \ln^3(2 + \|u\|_{H^{3/2}(\mathbb{R})}).
\end{aligned}$$

Notice also that the second and third term involved in the energy $\mathcal{E}(u)$ can be estimated as follows:

$$\|u\|_{H^1(\mathbb{R})}^2 \|u\|_{L^\infty(\mathbb{R})}^3 \leq C \|u\|_{H^{3/2}(\mathbb{R})} \ln^{3/2}(2 + \|u\|_{H^{3/2}(\mathbb{R})}).$$

By combining the estimate above with (4.2) and (4.1), and by recalling that $\frac{d}{dt} \|u\|_{L^2(\mathbb{R})}^2 = 0$ we deduce

$$\begin{aligned}
& \|u(t)\|_{H^{3/2}(\mathbb{R})}^2 \leq C \|u(0)\|_{H^{3/2}(\mathbb{R})}^2 + C \sup_{s \in (0, t)} \|u(s)\|_{H^{3/2}(\mathbb{R})} \ln^{3/2}(2 + \|u(s)\|_{H^{3/2}(\mathbb{R})}) \\
& \quad + C \int_0^t \|u(s)\|_{H^{3/2}(\mathbb{R})}^2 \ln(2 + \|u(s)\|_{H^{3/2}(\mathbb{R})}) + \|u(s)\|_{H^{3/2}(\mathbb{R})} \ln^3(2 + \|u(s)\|_{H^{3/2}(\mathbb{R})}) ds.
\end{aligned}$$

We conclude by a suitable version of the Gronwall Lemma (see [13] for more details on this point). \square

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J. BELLAZZINI,
UNIVERSITÀ DI SASSARI, VIA PIANDANNA 4, 70100 SASSARI, ITALY
E-mail address: jbellazzini@uniss.it

V. GEORGIEV
DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DI PISA LARGO B. PONTECORVO 5, 56100
PISA, ITALY
E-mail address: georgiev@dm.unipi.it

N. VISCIGLIA,
DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DI PISA LARGO B. PONTECORVO 5, 56100
PISA, ITALY
E-mail address: viscigli@dm.unipi.it